Lecture 17: Convergence of Fourier Series

- We will establish criteria for 3 types of convergence:

1.) Pointwise Convergence: $\{f_n\} \to f$ pointwise on Ω if for every $x \in \Omega$, $\{f_n(x)\} \to f(x)$

e.g.) $\{\frac{1}{x}n\} \to 0$ pointwise on $(0,\infty) \subseteq \mathbb{R}$

2.) Uniform Convergence: $\{f_n\} \to f$ uniformly on Ω if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n > N$,

$$\sup_{x \in \Omega} |f_n(x) - f(x)| < \varepsilon$$

e.g.) $(x + 1/n)^2 \to x^2$ uniformly on $(0,1)$

3.) L^2 Convergence: Convergence in L^2 norm.

> Notice, $\frac{1}{xn}$ fails to converge uniformly on $(0,\infty)$

Pointwise Convergence

Thm 8.3 Suppose $f \in L^2(\mathbb{T})$ and that for $x \in \mathbb{T}$,

$$\operatorname*{ess\text{-}sup}_{y \in [-\varepsilon, \varepsilon]} \left| \frac{f(x) - f(x-y)}{y} \right| < \infty$$

holds for some $\varepsilon > 0$. Then, $\lim_{n \to \infty} S_n[f](x) = f(x)$.

Rmk: If $f \in C^1(\mathbb{T})$, $\left| \frac{f(x) - f(x-y)}{y} \right| = \left| \frac{1}{y} \int_{x-y}^{x} f'(t)\,dt \right| \leq \|f'\|_\infty$

and this holds automatically.

$\rightsquigarrow$ There are counterexamples for $f \in C^0$ only.

Aside: The Dirichlet Kernel

- we manipulate the partial sums to gain a useful tool

$$S_n[f](x) = \sum_{-n}^{n} e^{ikx} \cdot \frac{1}{2\pi} \int_0^{2\pi} f(y) e^{-iky}\,dy$$

$$= \int_0^{2\pi} f(y) \left[\sum_{-n}^{n} \frac{1}{2\pi} e^{ik(x-y)} \right] dy$$

$$= \int_0^{2\pi} f(y) D_n(x-y)\,dy \quad \text{for } D_n(x) = \frac{1}{2\pi} \sum_{k=-n}^{n} e^{ikx}$$

- In previous notation, $S_n[f] = f * D_n$

⇝ Notice $D_n(x)$ is smooth & $\int_0^{2\pi} D_n(x)dx = 1$

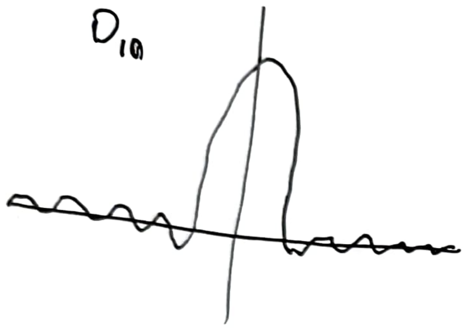
- We may simplify further. Recall $1 + z + \dots + z^m = \frac{z^{m+1} - 1}{z - 1}$

So for $z = e^{it}$

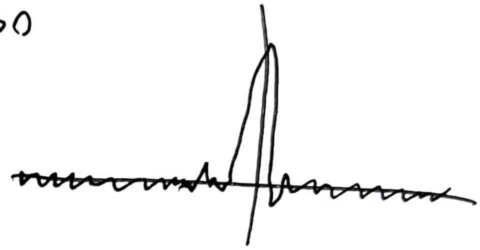
$$D_n(t) = \frac{1}{2\pi} \frac{e^{i(n+1)t} - e^{-int}}{e^{it} - 1} = \frac{1}{2\pi} \cdot \frac{\sin((n+\frac{1}{2})t)}{\sin(\frac{1}{2}t)} \quad (B)$$

- D_n:

D_{10}



D_{50}



⇝ Acts similarly to the heat kernel

Pf Let us rewrite $S_n[f](x) = \int_0^{2\pi} D_n(y) f(x-y)dy$.

Then, $f(x) - S_n[f](x) = \int_0^{2\pi} D_n(y)[f(x) - f(x-y)]dy$ → f periodic & (B) ⇒

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(x) - f(x-y)}{e^{iy} - 1} [e^{i(n+1)y} - e^{-iny}]dy$$

- Set $h(y) = \frac{f(x) - f(x-y)}{e^{iy} - 1} = \underbrace{\frac{f(x) - f(x-y)}{y}}_{C} \cdot \underbrace{\frac{y}{e^{iy} - 1}}_{D}$

Notice (C) is bdd. by assumption for small y.

Since $e^{iy} - 1 = \sum_{n=1}^{\infty} (iy)^n/n!$, $(D) = \frac{y}{iy - y^2/2 + \dots}$

is bdd as well as $y \to 0$.

•) Hence, $h(y)$ is bounded for $y \in [-\varepsilon, \varepsilon]$.

Since $f \in L^2(\mathbb{T})$ & $(e^{iy}-1)^{-1}$ is bounded for $y \in \pm[\varepsilon, \pi]$, $h \in L^2(\mathbb{T})$ as well.

Then, if $C_n[h]$ is the n^{th} Fourier Coefficient of h,

$$f(x) - S_n[f](x) = C_{n-1}[h] - C_n[h]$$

By Bessel's Inequality, the RHS $\to 0$ as $n \to \infty$. □
$\left(\sum_{-\infty}^{\infty} C_n[h]^2 < \infty \right)$

Uniform Convergence

• Why is uniform convergence important? First, note that uniform convergence implies pointwise. Second, note that Pointwise convergence doesn't preserve continuity:

$$\{ e^{-nx^2} \} \to \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} \quad \text{pointwise!}$$

However...

Lemma 8.4 Suppose $\{f_n\} \subset C^0(\Omega)$ for a domain $\Omega \subseteq \mathbb{R}^?$. If $\{f_n\} \to f : \Omega \to \mathbb{R}$ uniformly, then $f \in C^0(\Omega)$.

Pf Fix $x \in \Omega$ and pick $\varepsilon > 0$. First, there exists $N \in \mathbb{N}$ ~~second~~, for $n > N$.
So $\sup_{y \in \Omega} |f_n(y) - f(y)| < \varepsilon/3$.

Pick one such n & fix it. Second, there exists $\delta > 0$ so $|x-y| < \delta$ implies $|f_n(x) - f_n(y)| < \varepsilon/3$.

Then, for $|x-y| < \delta$,

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3$$
$$< \varepsilon.$$ □

Thm 8.5 For $f \in C^1(\mathbb{T})$, $S_n[f] \to f$ uniformly.

Pf Notice that $f' \in C^0(\mathbb{T})$, so integrating by parts gives

$$c_k[f'] = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(y)e^{-iky}dy = \frac{1}{2\pi}f(y)e^{-iky}\Big|_{-\pi}^{\pi} + \frac{ik}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iky}dy$$

or $c_k[f'] = ik\, c_k[f]$

As $f' \in \mathcal{L}^2(\mathbb{T})$, $\sum_{k\in\mathbb{Z}} |k\, c_k[f]|^2 < \infty$ by Bessel's Ineq.

Next, Consider $a_k = |k\, c_k[f]|$ and $a = \{a_k\}_{k\in\mathbb{Z}/\{0\}}$ has $a \in \ell^2$. Set $b_k = 1/k$ so $b = \{b_k\} \in \ell^2$ as well. By Cauchy-Schwarz

$$\langle a, b\rangle_{\ell^2} \le \|a\|_{\ell^2}\|b\|_{\ell^2} < \infty$$

or $$\sum_{k\in\mathbb{Z}} |c_k[f]| < \infty \qquad (E)$$

- By our pointwise convergence thm, $f \in C^1$ gives for each $x \in \mathbb{T}$, $f(x) = \sum_{-\infty}^{\infty} c_n[f]\phi_n(x)$

and

$$|S_n[f](x) - f(x)| \le \sum_{|k|>n} |c_n[f]| \cdot 1 \qquad (|\phi_n| = 1)$$

by (E), the RHS $\to 0$ as $n \to \infty$. Since it has no dependence on x, $S_n[f] \to f$ uniformly. ☐

Convergence in L^2

- The uniform convergence above on $\mathbb{T}$ implies L^2 convergence:

$$\|f_n - f\|_2^2 = \int_{-\pi}^{\pi} |f_n - f|^2 dx \leq 2\pi \sup_{x \in \mathbb{T}} |f_n(x) - f(x)|^2$$

So that we have convergence in L^2 for C^1 functions. We extend to all of L^2.

Thm 8.6 The normalized periodic Fourier eigenfunctions

$$\phi_k = 1/\sqrt{2\pi}\, e^{ikx} \qquad k \in \mathbb{Z}$$

form an orthonormal basis for $L^2(\mathbb{T})$.

Pf Suppose $\langle u, \phi_k \rangle = 0$ for all k. If we show $u = 0$, then $\{\phi_k\}$ is a basis by thm. 7.10.

To do this, we apply that C_c^∞ is dense in L^2. As we noted above, $S_n[\psi] \to \psi$ in L^2 for any $\psi \in C^1$, so

$$\langle u, \psi \rangle = \lim_{n\to\infty} \langle u, S_n[\psi] \rangle = \lim_{n\to\infty} 0 = 0$$

by the orthogonality assumption. We may pick some $\{\psi_n\} \subset C_c^\infty$, $\psi_n \to u$ in L^2, and $\langle u, u \rangle = \lim_{n\to\infty} \langle u, \psi_n \rangle = 0$, so $u = 0$. □

Corollary: Parseval's Identity

For $f \in L^2(\mathbb{T})$, the periodic Fourier coefficients $c_k[f]$ satisfy

$$\sum_{k \in \mathbb{Z}} |c_k[f]|^2 = \frac{1}{2\pi} \|f\|_2$$

Pf Bessel's Inequality & the above □

Rmk: This implies $\langle f, g \rangle = 2\pi \sum_{k \in \mathbb{Z}} c_k[f] \overline{c_k[g]}$.

Ex.) In the case $h(x) = \begin{cases} 0 & x \in [0, \pi/2) \\ 1 & x \in [\pi/2, \pi] \end{cases}$ we computed $c_k[h] = \pm \frac{1}{\pi k}$ for k odd, $c_0[h] = 1/2$, & $c_k[h] = 0$ otherwise.

Then,

$$\sum_{k \in \mathbb{Z}} |c_k[h]|^2 = \frac{1}{4} + 2 \sum_{k \in \mathbb{N} \text{ odd}} \frac{1}{\pi^2 k^2}$$

Alternately, $\|h_2\|^2 = \pi$

So Parseval $\Rightarrow$

$$\frac{1}{4} + \frac{2}{\pi^2} \sum_{\substack{k \in \mathbb{N} \\ k \text{ odd}}} 1/k^2 = 1/2$$

or
$$\sum_{\substack{k \in \mathbb{N} \\ k \text{ odd}}} 1/k^2 = \pi^2/8$$